Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Applied Mathematics and Mechanics 70 (2006) 102–115
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Asymptotic solutions of non-classical boundary-value problems of the natural vibrations of orthotropic shells☆

L.A. Agalovyan, L.G. Gulgazaryan

Yerevan, Armenia

Received 24 June 2004

Abstract

The natural vibrations of orthotropic shells are considered in a three-dimensional formulation for different versions of the boundary conditions on the faces: rigid clamping rigid clamping, rigid clamping free surface, and mixed conditions. Asymptotic solutions of the corresponding dynamic equations of the three-dimensional problem of the theory of elasticity are obtained. The principal values of the frequencies of natural vibrations are determined. It is shown that three types of natural vibrations occur in the shell: two shear vibrations and a longitudinal vibration, which are due solely to the boundary conditions on the faces. It is proved that each boundary layer has its own natural frequency. The boundary-layer functions are determined and the rates at which they decrease with distance from the faces inside the shell are established.

A number of publications[1–4] are devoted to the asymptotic method of solving the classical static boundary-value problem for plates and shells (appropriate components of the stress tensor are specified on the faces). The classical problem of the natural and forced vibrations of shells are considered by the same method in Refs. 5–8. Another branch of the asymptotic method was developed and an asymptotic theory of anisotropic plates and shells was constructed in Ref. 9. The method turned out to be particularly effective for solving non-classical boundary-value problems of thin bodies (values of the displacement vector or mixed conditions were specified on the faces). A characteristic feature of these problems is the fact that, when solving them, the hypotheses and assumptions of the classical theory are inapplicable. An essentially new asymptotic representation was established for the components of the stress tensor and the displacement vector which enables a solution of the corresponding three-dimensional problem of the theory of elasticity to be obtained with an asymptotic accuracy specified in advance.[10–14] Some non-classical problems of the natural and forced vibrations of thin bodies have also been solved in Refs. 15–19; a review of research in this area can be found in Ref. 20.

1. Fundamental equations and formulation of the boundary-value problems

Consider the natural vibrations of an orthotropic shell of thickness $2h$: $\Omega = \{\alpha, \beta, \gamma; \alpha, \beta \in \Omega_0, -h \le \gamma \le h\}$, where Ω_0 is the middle surface, α and β are the lines of curvature of the middle surface of the shell, and γ is a rectilinear axis, directed perpendicular to the middle surface. It is required to obtain the non-zero solutions of the dynamic equations

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 1, pp. 111–125, 2006.

E-mail addresses: aghal@mechins.sci.am, lusina@mail.ru (L.A. Agalovyan).

0021-8928/$ – see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2006.03.005

of the theory of elasticity in the chosen triorthogonal system of coordinates for a series of boundary conditions on the faces $\gamma = \pm h$. To simplify the calculations we will use the components of the asymmetric stress tensor τ_{ij}.[1,9,21]

We have the following equations of motion

$$\frac{1}{AB}\frac{\partial}{\partial\alpha}(B\tau_{\alpha\alpha}) - k_\beta\tau_{\beta\beta} + \frac{1}{AB}\frac{\partial}{\partial\beta}(A\tau_{\beta\alpha}) + k_\alpha\tau_{\alpha\beta} + \left(1+\frac{\gamma}{R_1}\right)\frac{\partial\tau_{\alpha\gamma}}{\partial\gamma} + \frac{2\tau_{\alpha\gamma}}{R_1} = \rho\frac{\partial^2 U}{\partial t^2}$$

$$(A \leftrightarrow B;\ \alpha \leftrightarrow \beta;\ R_1, R_2;\ U, V) \tag{1.1}$$

$$\frac{\partial\tau_{\gamma\gamma}}{\partial\gamma} - \left(\frac{\tau_{\alpha\alpha}}{R_1} + \frac{\tau_{\beta\beta}}{R_2}\right) + \frac{1}{A}\frac{\partial\tau_{\alpha\gamma}}{\partial\alpha} + \frac{1}{B}\frac{\partial\tau_{\beta\gamma}}{\partial\beta} + k_\beta\tau_{\alpha\gamma} + k_\alpha\tau_{\beta\gamma} = \rho\frac{\partial^2 W}{\partial t^2}$$

$$\left(1+\frac{\gamma}{R_1}\right)\tau_{\alpha\beta} = \left(1+\frac{\gamma}{R_2}\right)\tau_{\beta\alpha} \text{ (the symmetry condition),}$$

and the following equations of state (the elasticity relations)

$$\left(1+\frac{\gamma}{R_2}\right)\left(\frac{1}{A}\frac{\partial U}{\partial\alpha} + k_\alpha V + \frac{W}{R_1}\right) = \left(1+\frac{\gamma}{R_1}\right)a_{11}\tau_{\alpha\alpha} + \left(1+\frac{\gamma}{R_2}\right)a_{12}\tau_{\beta\beta} + a_{13}\tau_{\gamma\gamma}$$

$$(A, B;\ \alpha \leftrightarrow \beta;\ R_1 \leftrightarrow R_2;\ U \leftrightarrow V;\ a_{11}, a_{22};\ a_{13}, a_{23})$$

$$\left[1+\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{\gamma^2}{R_1R_2}\right]\frac{\partial W}{\partial\gamma} = \left(1+\frac{\gamma}{R_1}\right)a_{13}\tau_{\alpha\alpha} + \left(1+\frac{\gamma}{R_2}\right)a_{23}\tau_{\beta\beta} + a_{33}\tau_{\gamma\gamma}$$

$$\left(1+\frac{\gamma}{R_1}\right)\left(\frac{1}{B}\frac{\partial U}{\partial\beta} - k_\beta V\right) + \left(1+\frac{\gamma}{R_2}\right)\left(\frac{1}{A}\frac{\partial V}{\partial\alpha} - k_\alpha U\right) = \left(1+\frac{\gamma}{R_1}\right)a_{66}\tau_{\alpha\beta} \tag{1.2}$$

$$\left[1+\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{\gamma^2}{R_1R_2}\right]\frac{\partial U}{\partial\gamma} - \left(1+\frac{\gamma}{R_2}\right)\frac{U}{R_1} + \frac{1}{A}\left(1+\frac{\gamma}{R_2}\right)\frac{\partial W}{\partial\alpha} = \left(1+\frac{\gamma}{R_1}\right)a_{55}\tau_{\alpha\gamma}$$

$$(A, B;\ \alpha, \beta;\ R_1 \leftrightarrow R_2;\ U, V;\ a_{55}, a_{44})$$

where k_α and k_β are the geodesic curvatures, A and B are coefficients of the first quadratic form, R_1 and R_2 are the principal radii of curvature of the middle surface, ρ is the density and a_{ij} are the constants of elasticity ($a_{ij} = a_{ji}$).

One of the following groups of conditions is specified on the faces $\gamma = h$

$$\tau_{\alpha\gamma}(h) = 0, \quad \tau_{\beta\gamma}(h) = 0, \quad \tau_{\gamma\gamma}(h) = 0 \tag{1.3}$$

or

$$U(h) = 0, \quad V(h) = 0, \quad W(h) = 0, \tag{1.4}$$

while one of the following groups of conditions is specified on the surface $\gamma = -h$

$$U(-h) = 0, \quad V(-h) = 0, \quad W(-h) = 0 \tag{1.5}$$

or

$$\tau_{\alpha\gamma}(-h) = 0, \quad \tau_{\beta\gamma}(-h) = 0, \quad W(-h) = 0 \tag{1.6}$$

The conditions on the side surface will not be specified for the time being. They affective the values of the amplitudes of the vibrations in the boundary layer.

2. Solution of the internal problem

We will change in Eqs. (1.1) and (1.2) to dimensionless coordinates and displacements in accordance with the formulae

$$\alpha = R\xi, \quad \beta = R\eta, \quad \gamma = \varepsilon R\zeta = h\zeta, \quad U = Ru, \quad V = Rv, \quad W = Rw$$

where R is a characteristic dimension of the shell (the least of the radii of curvature and of the linear dimensions of the middle surface), and $\varepsilon = h/R$ is a small parameter. The solution of the transformed equations will be sought in the form

$$Q_{\alpha\beta} = Q_{jk}(\xi, \eta, \zeta)e^{i\omega t} \ (\alpha, \beta, \gamma); \quad j, k = 1, 2, 3 \tag{2.1}$$

where $Q_{\alpha\beta}$ is any of the values of the stresses and displacements and ω is the frequency of natural vibrations. As a result we obtain a system in Q_{jk}, singularly perturbed by the small parameter ε

$$\begin{gathered}
\frac{1}{AB}\frac{\partial}{\partial\xi}(B\tau_{11}) - k_\beta R\tau_{22} + \frac{1}{AB}\frac{\partial}{\partial\eta}(A\tau_{21}) + k_\alpha R\tau_{12} + (\varepsilon^{-1} + r_1\zeta)\frac{\partial\tau_{13}}{\partial\zeta} + 2r_1\tau_{13} = -\varepsilon^{-2}\omega_*^2 u \\
(A \leftrightarrow B;\ \alpha \leftrightarrow \beta;\ r_1, r_2;\ \xi \leftrightarrow \eta;\ u, \upsilon;\ \tau_{11} \leftrightarrow \tau_{22};\ \tau_{12} \leftrightarrow \tau_{21};\ \tau_{13}, \tau_{23})
\end{gathered}$$

$$\begin{gathered}
\varepsilon^{-1}\frac{\partial\tau_{33}}{\partial\zeta} - (r_1\tau_{11} + r_2\tau_{22}) + \frac{1}{A}\frac{\partial\tau_{13}}{\partial\xi} + \frac{1}{B}\frac{\partial\tau_{23}}{\partial\eta} + k_\beta R\tau_{13} + k_\alpha R\tau_{23} = -\varepsilon^{-2}\omega_*^2 w \\
(1 + \varepsilon r_2\zeta)\left(\frac{1}{A}\frac{\partial u}{\partial\xi} + k_\alpha R\upsilon + r_1 w\right) = (1 + \varepsilon r_1\zeta)a_{11}\tau_{11} + (1 + \varepsilon r_2\zeta)a_{12}\tau_{22} + a_{13}\tau_{33} \\
(A, B;\ \alpha, \beta;\ r_1 \leftrightarrow r_2;\ \xi, \eta;\ u \leftrightarrow \upsilon;\ \tau_{11} \leftrightarrow \tau_{22};\ a_{11}, a_{22};\ a_{13}, a_{23}) \\
[\varepsilon^{-1} + \zeta(r_1 + r_2) + \varepsilon\zeta^2 r_1 r_2]\frac{\partial w}{\partial\zeta} = (1 + \varepsilon r_1\zeta)a_{13}\tau_{11} + (1 + \varepsilon r_2\zeta)a_{23}\tau_{22} + a_{33}\tau_{33} \\
(1 + \varepsilon r_1\zeta)\left(\frac{1}{B}\frac{\partial u}{\partial\eta} - k_\beta R\upsilon\right) + (1 + \varepsilon r_2\zeta)\left(\frac{1}{A}\frac{\partial\upsilon}{\partial\xi} - k_\alpha R u\right) = (1 + \varepsilon r_1\zeta)a_{66}\tau_{12} \\
[\varepsilon^{-1} + \zeta(r_1 + r_2) + \varepsilon\zeta^2 r_1 r_2]\frac{\partial u}{\partial\zeta} - (1 + \varepsilon r_2\zeta)r_1 u + \frac{1}{A}(1 + \varepsilon r_2\zeta)\frac{\partial w}{\partial\xi} = (1 + \varepsilon r_1\zeta)a_{55}\tau_{13} \\
(A, B;\ r_1 \leftrightarrow r_2;\ \xi, \eta;\ u, \upsilon;\ \tau_{13}, \tau_{23};\ a_{55}, a_{44}) \\
(1 + \varepsilon r_1\zeta)\tau_{12} = (1 + \varepsilon r_2\zeta)\tau_{21} \\
r_1 = \frac{R}{R_1}, \quad r_2 = \frac{R}{R_2}, \quad \omega_*^2 = \rho h^2\omega^2
\end{gathered} \tag{2.2}$$

The solution of system (2.2) is the sum of the solutions of the internal problem and of the boundary layer.[1,9] The solution of the internal problem will be sought in the form of the asymptotic representation[9,17]

$$\begin{gathered}
\tau_{jk}(\xi, \eta, \zeta) = \varepsilon^{-1+s}\tau_{jk}^{(s)}(\xi, \eta, \zeta), \quad j, k = 1, 2, 3; \quad s = \overline{0, N} \\
(u(\xi, \eta, \zeta), \upsilon(\xi, \eta, \zeta), w(\xi, \eta, \zeta)) = \varepsilon^s(u^{(s)}(\xi, \eta, \zeta), \upsilon^{(s)}(\xi, \eta, \zeta), w^{(s)}(\xi, \eta, \zeta)) \\
\omega_* = \varepsilon^s\omega_*^{(s)}
\end{gathered} \tag{2.3}$$

Here and henceforth $s = \overline{0, N}$ denotes that summation in the limits 0, N is carried out with respect to the dummy (repeated) index s.

It follows from the asymptotic form (2.3) that, unlike the classical theory,[1,9] for this class of problems all the components of the stress tensor are asymptotically equally justified, the displacements are also equally justified (are of the same order), and the assumptions of the classical theory of plates and shells are inapplicable here.

Substituting expressions (2.3) into system (2.2) and applying Cauchy's rule of the multiplication of series, to determine the unknown expansion coefficients $Q_{jk}^{(s)}$ we obtain the consistent system

$$\begin{aligned}
&\frac{1}{AB}\frac{\partial}{\partial\xi}(B\tau_{11}^{(s-1)}) - k_\beta R\tau_{22}^{(s-1)} + \frac{1}{AB}\frac{\partial}{\partial\eta}(A\tau_{21}^{(s-1)}) + k_\alpha R\tau_{12}^{(s-1)} + \frac{\partial\tau_{13}^{(s)}}{\partial\zeta} + r_1\zeta\frac{\partial\tau_{13}^{(s-1)}}{\partial\zeta} + \\
&+ 2r_1\tau_{13}^{(s-1)} + c^{(m)}u^{(s-m)} = 0, \quad m = \overline{0, s} \\
&(A \leftrightarrow B;\ \alpha \leftrightarrow \beta;\ r_1, r_2;\ \xi \leftrightarrow \eta;\ u, \upsilon;\ \tau_{12} \leftrightarrow \tau_{21};\ \tau_{11} \leftrightarrow \tau_{22};\ \tau_{13}, \tau_{23}) \\
&\frac{\partial\tau_{33}^{(s)}}{\partial\zeta} - r_1\tau_{11}^{(s-1)} - r_2\tau_{22}^{(s-1)} + \frac{1}{A}\frac{\partial\tau_{13}^{(s-1)}}{\partial\xi} + \frac{1}{B}\frac{\partial\tau_{23}^{(s-1)}}{\partial\eta} + k_\beta R\tau_{13}^{(s-1)} + k_\alpha R\tau_{23}^{(s-1)} + c^{(m)}w^{(s-m)} = 0 \\
&\frac{1}{A}\frac{\partial u^{(s-1)}}{\partial\xi} + k_\alpha R\upsilon^{(s-1)} + r_1 w^{(s-1)} + r_2\zeta\left(\frac{1}{A}\frac{\partial u^{(s-2)}}{\partial\xi} + k_\alpha R\upsilon^{(s-2)} + r_1 w^{(s-2)}\right) = \\
&= r_1\zeta a_{11}\tau_{11}^{(s-1)} + r_2\zeta a_{12}\tau_{22}^{(s-1)} + \sum_1^{(s)} \\
&\left(A, B;\ \alpha, \beta;\ r_1 \leftrightarrow r_2;\ \xi, \eta;\ u \leftrightarrow \upsilon;\ \tau_{11} \leftrightarrow \tau_{22};\ a_{11}, a_{22};\ \sum_1^{(s)}, \sum_2^{(s)}\right) \\
&\frac{\partial w^{(s)}}{\partial\zeta} + \zeta(r_1 + r_2)\frac{\partial w^{(s-1)}}{\partial\zeta} + \zeta^2 r_1 r_2\frac{\partial w^{(s-2)}}{\partial\zeta} = \sum_3^{(s)} + r_1\zeta a_{13}\tau_{11}^{(s-1)} + r_2\zeta a_{23}\tau_{22}^{(s-1)} \\
&\frac{1}{B}\frac{\partial u^{(s-1)}}{\partial\eta} - k_\beta R\upsilon^{(s-1)} + r_1\zeta\left(\frac{1}{B}\frac{\partial u^{(s-2)}}{\partial\eta} - k_\beta R\upsilon^{(s-2)}\right) + \frac{1}{A}\frac{\partial\upsilon^{(s-1)}}{\partial\xi} - k_\alpha R u^{(s-1)} + \\
&+ r_2\zeta\left(\frac{1}{A}\frac{\partial\upsilon^{(s-2)}}{\partial\xi} - k_\alpha R u^{(s-2)}\right) = a_{66}\tau_{12}^{(s)} + r_1\zeta a_{66}\tau_{12}^{(s-1)} \\
&\frac{\partial u^{(s)}}{\partial\zeta} + \zeta(r_1 + r_2)\frac{\partial u^{(s-1)}}{\partial\zeta} + \zeta^2 r_1 r_2\frac{\partial u^{(s-2)}}{\partial\zeta} - r_1 u^{(s-1)} - \zeta r_1 r_2 u^{(s-2)} + \frac{1}{A}\frac{\partial w^{(s-1)}}{\partial\xi} + \frac{r_2\zeta}{A}\frac{\partial w^{(s-2)}}{\partial\xi} = \\
&= a_{55}\tau_{13}^{(s)} + r_1\zeta a_{55}\tau_{13}^{(s-1)}\ (A, B;\ r_1 \leftrightarrow r_2;\ \xi, \eta;\ u, \upsilon;\ \tau_{13}, \tau_{23};\ a_{55}, a_{44}) \\
&\tau_{12}^{(s)} + r_1\zeta\tau_{12}^{(s-1)} = \tau_{21}^{(s)} + r_2\zeta_{21}^{(s-1)} \\
&c^{(m)} = \sum_{n=0}^{m}\omega_{*(m-n)}\omega_{*(n)}, \quad \sum_i^{(s)} = a_{i1}\tau_{11}^{(s)} + a_{i2}\tau_{22}^{(s)} + a_{i3}\tau_{33}^{(s)}
\end{aligned} \tag{2.4}$$

We note that a consistent system for $Q_{jk}^{(s)}$ can only be obtained for the asymptotic form (2.3).
System (2.4) can be written in the form

$$\begin{aligned}
&\tau_{12}^{(s)} = P_{1\tau}^{(s-1)}, \quad \tau_{21}^{(s)} = P_{1\tau}^{(s-1)} - r_2\zeta\tau_{21}^{(s-1)} + r_1\zeta\tau_{12}^{(s-1)}, \quad \sum_1^{(s)} = P_{2\tau}^{(s-1)}, \quad \sum_2^{(s)} = P_{3\tau}^{(s-1)} \\
&\frac{\partial\tau_{13}^{(s)}}{\partial\zeta} + c^{(m)}u^{(s-m)} = P_{6\tau}^{(s-1)}, \quad m = \overline{0, s} \quad (13, 23, 33;\ u, \upsilon, w;\ 6\tau, 5\tau, 4\tau) \\
&\frac{\partial u^{(s)}}{\partial\zeta} - a_{55}\tau_{13}^{(s)} = P_u^{(s-1)}, \quad \frac{\partial\upsilon^{(s)}}{\partial\zeta} - a_{44}\tau_{23}^{(s)} = P_\upsilon^{(s-1)}, \quad \frac{\partial w^{(s)}}{\partial\zeta} - \sum_3^{(s)} = P_w^{(s-1)}
\end{aligned} \tag{2.5}$$

where

$$P_{1\tau}^{(s-1)} = \frac{1}{a_{66}}\left[\frac{1}{B}\frac{\partial u^{(s-1)}}{\partial \eta} - k_\beta R \upsilon^{(s-1)} + r_1\zeta\left(\frac{1}{B}\frac{\partial u^{(s-2)}}{\partial \eta} - k_\beta R\upsilon^{(s-2)}\right) + \right.$$
$$\left. + \frac{1}{A}\frac{\partial \upsilon^{(s-1)}}{\partial \xi} - k_\alpha R u^{(s-1)} + r_2\zeta\left(\frac{1}{A}\frac{\partial \upsilon^{(s-2)}}{\partial \xi} - k_\alpha R u^{(s-2)}\right) - r_1\zeta a_{66}\tau_{12}^{(s-1)}\right]$$

$$P_{2\tau}^{(s-1)} = \frac{1}{A}\frac{\partial u^{(s-1)}}{\partial \xi} + k_\alpha R\upsilon^{(s-1)} + r_1 w^{(s-1)} + r_2\zeta\left(\frac{1}{A}\frac{\partial u^{(s-2)}}{\partial \xi} + k_\alpha R\upsilon^{(s-2)} + r_1 w^{(s-2)}\right) -$$
$$- r_1\zeta a_{11}\tau_{11}^{(s-1)} - r_2\zeta a_{12}\tau_{22}^{(s-1)}$$
$$(2\tau, 3\tau; A, B; \alpha, \beta; r_1 \leftrightarrow r_2; \xi, \eta; u \leftrightarrow \upsilon; \tau_{11} \leftrightarrow \tau_{22}; a_{11}, a_{22})$$
$$P_{4\tau}^{(s-1)} = r_1\tau_{11}^{(s-1)} + r_2\tau_{22}^{(s-1)} - \frac{1}{A}\frac{\partial \tau_{13}^{(s-1)}}{\partial \xi} - \frac{1}{B}\frac{\partial \tau_{23}^{(s-1)}}{\partial \eta} - k_\beta R\tau_{13}^{(s-1)} - k_\alpha R\tau_{23}^{(s-1)}$$
$$P_{5\tau}^{(s-1)} = -\frac{1}{AB}\frac{\partial}{\partial \eta}(A\tau_{22}^{(s-1)}) + k_\alpha R\tau_{11}^{(s-1)} - \frac{1}{AB}\frac{\partial}{\partial \xi}(B\tau_{12}^{(s-1)}) - k_\beta R\tau_{21}^{(s-1)} - r_2\zeta\frac{\partial \tau_{23}^{(s-1)}}{\partial \zeta} - 2r_2\tau_{23}^{(s-1)} \qquad (2.6)$$
$$(5\tau, 6\tau; A \leftrightarrow B; \alpha \leftrightarrow \beta; r_2, r_1; \xi \leftrightarrow \eta; \tau_{11} \leftrightarrow \tau_{22}; \tau_{12} \leftrightarrow \tau_{21}; \tau_{23}, \tau_{13})$$
$$P_u^{(s-1)} = -\zeta(r_1 + r_2)\frac{\partial u^{(s-1)}}{\partial \zeta} - \zeta^2 r_1 r_2\frac{\partial u^{(s-2)}}{\partial \zeta} + r_1 u^{(s-1)} + \zeta r_1 r_2 u^{(s-2)} - \frac{1}{A}\frac{\partial w^{(s-1)}}{\partial \xi} -$$
$$- \frac{r_2\zeta}{A}\frac{\partial w^{(s-2)}}{\partial \xi} + r_1\zeta a_{55}\tau_{13}^{(s-1)} \ (u, \upsilon; A, B; r_1 \leftrightarrow r_2; \xi, \eta; \tau_{13}, \tau_{23}; a_{55}, a_{44})$$
$$P_w^{(s-1)} = -\zeta(r_1 + r_2)\frac{\partial w^{(s-1)}}{\partial \zeta} - \zeta^2 r_1 r_2\frac{\partial w^{(s-2)}}{\partial \zeta} + r_1\zeta a_{13}\tau_{11}^{(s-1)} + r_2\zeta a_{23}\tau_{22}^{(s-1)}$$

Using relations (2.5), the components of the stress tensor can be expressed in terms of $u^{(s)}$, $\upsilon^{(s)}$, $w^{(s)}$.

$$\tau_{13}^{(s)} = \frac{1}{a_{55}}\left[\frac{\partial u^{(s)}}{\partial \zeta} - P_u^{(s-1)}\right], \quad \tau_{23}^{(s)} = \frac{1}{a_{44}}\left[\frac{\partial \upsilon^{(s)}}{\partial \zeta} - P_\upsilon^{(s-1)}\right]$$
$$\tau_{12}^{(s)} = P_{1\tau}^{(s-1)}, \quad \tau_{21}^{(s)} = P_{1\tau}^{(s-1)} - r_2\zeta\tau_{21}^{(s-1)} + r_1\zeta\tau_{12}^{(s-1)} \qquad (2.7)$$
$$\tau_{11}^{(s)} = \frac{1}{\Delta}\left[\Delta_2\frac{\partial w^{(s)}}{\partial \zeta} + \Delta_{23}P_{2\tau}^{(s-1)} + \Delta_1 P_{3\tau}^{(s-1)} - \Delta_2 P_w^{(s-1)}\right]$$
$$(11, 22, 33; \Delta_2, \Delta_3, \Delta_{12}; \Delta_{23}, \Delta_1, \Delta_2; \Delta_1, \Delta_{13}, \Delta_3)$$

where

$$\Delta_1 = a_{13}a_{23} - a_{33}a_{12}, \quad \Delta_2 = a_{12}a_{23} - a_{22}a_{13}, \quad \Delta_3 = a_{13}a_{12} - a_{11}a_{23}$$
$$\Delta_{ij} = a_{ii}a_{jj} - a_{ij}^2, \quad i, j = 1, 2, 3; \quad \Delta = a_{11}\Delta_{23} + a_{13}\Delta_2 + a_{12}\Delta_1 \qquad (2.8)$$

while to determine the components of the displacement vector we obtain the equations

$$\frac{\partial^2 u^{(s)}}{\partial\zeta^2} + a_{55}c^{(m)}u^{(s-m)} = a_{55}P_{6\tau}^{(s-1)} + \frac{\partial P_u^{(s-1)}}{\partial\zeta}, \quad m = \overline{0, s} \quad (u, \upsilon; a_{55}, a_{44}; 6\tau, 5\tau)$$
$$\frac{\partial^2 w^{(s)}}{\partial\zeta^2} + \frac{\Delta}{\Delta_{12}}c^{(m)}w^{(s-m)} = F_w^{(s-1)} \tag{2.9}$$

$$F_w^{(s-1)} = \frac{1}{\Delta_{12}}\left[\Delta P_{4\tau}^{(s-1)} - \Delta_2\frac{\partial P_{2\tau}^{(s-1)}}{\partial\zeta} - \Delta_3\frac{\partial P_{3\tau}^{(s-1)}}{\partial\zeta} + \Delta_{12}\frac{\partial P_w^{(s-1)}}{\partial\zeta}\right]$$

When $s = 0$ system (2.9) is converted into a system of three independent equations

$$\frac{\partial^2 u^{(0)}}{\partial\zeta^2} + (\chi^u)^2 u^{(0)} = 0, \ (u, \upsilon, w); \ \chi^u = \sqrt{a_{55}}\omega_{*0}, \ \chi^\upsilon = \sqrt{a_{44}}\omega_{*0}, \ \chi^w = \sqrt{\frac{\Delta}{\Delta_{12}}}\omega_{*0} \tag{2.10}$$

the solutions of which have the form

$$u^{(0)}(\xi, \eta, \zeta) = C_1^{(0)}(\xi, \eta)\sin\chi^u\zeta + C_2^{(0)}(\xi, \eta)\cos\chi^u\zeta \ (u, \upsilon, w; 1, 3, 5; 2, 4, 6) \tag{2.11}$$

By substituting expressions (2.11) into relations (2.7) and satisfying boundary conditions (1.3) and (1.5), we obtain three independent homogeneous algebraic systems in the unknowns $C_i^{(0)}$. From the condition for non-zero solutions of these systems to exist we obtain the following frequency equations and the frequency values corresponding to them

$$\cos 2\chi^u = 0 \Rightarrow \chi_n^u = \sqrt{a_{55}}\omega_{*0}^u = \frac{\pi(2n+1)}{4}\left(u, \upsilon, w; \sqrt{a_{55}}, \sqrt{a_{44}}, \sqrt{\frac{\Delta}{\Delta_{12}}}\right) \tag{2.12}$$

Solution (2.11), taking relations (2.12) into account, can be written in the form

$$u_n^{(0,u)} = C_{2n}^{(0)}(\xi, \eta)\psi_n(\zeta), \quad \upsilon_n^{(0,\upsilon)} = C_{4n}^{(0)}(\xi, \eta)\psi_n(\zeta), \quad w_n^{(0,w)} = C_{6n}^{(0)}(\xi, \eta)\psi_n(\zeta)$$
$$\psi_n(\zeta) = \cos\frac{\pi(2n+1)}{4}(1-\zeta) \tag{2.13}$$

The coefficients $C_{in}^{(0)}(\xi, \eta)$ are found from the initial conditions by a generally known method. It is easy to show that the functions $\{\psi_n(\zeta)\}$ comprise an orthonormalized system in the section $[-1, 1]$. The frequencies (2.12) are identical with the frequencies of shear and longitudinal vibrations of orthotropic plates for similar boundary conditions (1.3) and (1.5).[17]

3. The contribution of the approximations $s \geq 1$

The solution when $s \geq 1$ will depend on which of the values of the frequencies ω_{*0n}^u, ω_{*0n}^υ, ω_{*0n}^w is taken as the basis for calculations, in particular, when solving Eq. (2.9). It is necessary to consider all three cases. When $s \geq 1$, Eq. (2.9) become inhomogeneous.

Consider the approximation $s = 1$. If $\omega_{*0} = \omega_{*0n}^u$, from relations (2.7) and (2.10) and the boundary conditions (1.3) and (1.5) for $\tau_{\beta\gamma}$, $\tau_{\gamma\gamma}$, V, W we obtain the equations

$$\upsilon_n^{(0,u)} = w_n^{(0,u)} = 0, \quad \tau_{23}^{(0,u)} = \tau_{33}^{(0,u)} = \tau_{12}^{(0,u)} = \tau_{21}^{(0,u)} = \tau_{11}^{(0,u)} = \tau_{22}^{(0,u)} = 0 \tag{3.1}$$

since, after satisfying these boundary conditions the algebraic systems of homogeneous equations obtained will have non-zero determinants in view of the fact that $\omega_{*0} = \omega_{*0n}^u$ is not a solution of the implicitly written Eq. (2.12).

System (2.9) can be converted to the following system of equations

$$\frac{\partial^2 u_n^{(1,u)}}{\partial \zeta^2} + a_{55}(\omega_{*0n}^u)^2 u_n^{(1,u)} + 2a_{55}\omega_{*0n}^u \omega_{*1n}^u u_n^{(0,u)} = F_u^{(0,u)} \tag{3.2}$$

$$\frac{\partial^2 \upsilon_n^{(1,u)}}{\partial \zeta^2} + a_{44}(\omega_{*0n}^u)^2 \upsilon_n^{(1,u)} = 0 \tag{3.3}$$

$$\frac{\partial^2 w_n^{(1,u)}}{\partial \zeta^2} + \frac{\Delta}{\Delta_{12}}(\omega_{*0n}^u)^2 w_n^{(1,u)} = F_w^{(0,u)} \tag{3.4}$$

where

$$F_u^{(0,u)} = -(r_1 + r_2)\left[\zeta \frac{\partial^2 u_n^{(0,u)}}{\partial \zeta^2} + \frac{\partial u_n^{(0,u)}}{\partial \zeta}\right]$$
$$F_w^{(0,u)} = \frac{1}{\Delta_{12}}\left[\Delta P_{4\tau}^{(0,u)} - \Delta_2 \frac{\partial P_{2\tau}^{(0,u)}}{\partial \zeta} - \Delta_3 \frac{\partial P_{3\tau}^{(0,u)}}{\partial \zeta} + \Delta_{12}\frac{\partial P_w^{(0,u)}}{\partial \zeta}\right] \tag{3.5}$$

It follows from Eq. (3.3), relations (2.7) and boundary conditions (1.3) and (1.5) that

$$\upsilon_n^{(1,u)} = 0, \quad \tau_{23}^{(1,u)} = 0 \tag{3.6}$$

The solution of Eq. (3.4) has the form

$$w_n^{(1,u)} = C_{5n}^{(1,u)} \sin \sqrt{\frac{\Delta}{\Delta_{12}}}\omega_{*0n}^u \zeta + C_{6n}^{(1,u)} \cos \sqrt{\frac{\Delta}{\Delta_{12}}}\omega_{*0n}^u \zeta + w_0^{(1,u)} \tag{3.7}$$

where $w_0^{(1,u)}$ is a particular solution of Eq. (3.4). By satisfying boundary conditions (1.3) and (1.5) for W and $\tau_{\gamma\gamma}$ and taking into account the fact that the determinant of the system obtained is non-zero, we can uniquely determine the unknown coefficients $C_{5n}^{(1,u)}$ and $C_{6n}^{(1,u)}$. Substituting the values of $u_n^{(0,u)}$ and $w_n^{(1,u)}$ into relation (2.7) we determine $\tau_{33}^{(1,u)}, \tau_{22}^{(1,u)}, \tau_{11}^{(1,u)}, \tau_{12}^{(1,u)}, \tau_{21}^{(1,u)}$.

The solution of Eq. (3.2) can be sought in the form of an expansion in an orthonormal system of functions,[22,23] for which we can take the eigenfunctions $\{\psi_n(\zeta)\}$.

We will expand the functions $u_n^{(1,u)}$ and $F_u^{(0,u)}$ in terms of these functions

$$u_n^{(1,u)}(\xi,\eta,\zeta) = \sum b_{1nm}(\xi,\eta)\psi_m(\zeta), \quad F_u^{(0,u)} = \sum f_{1m}^u \psi_m(\zeta)$$
$$f_{1m}^u = \int_{-1}^{1} F_u^{(0,u)}\psi_m(\zeta)d\zeta = -\frac{(r_1 + r_2)C_{2m}^{(0,u)}}{4} \tag{3.8}$$

Here and henceforth the summation sign denotes summation from $m = 0$ to $m = \infty$.

Boundary conditions (1.3) and (1.5) will be satisfied identically.

The functions $\{\psi_m(\zeta)\}$ satisfy the equation

$$\frac{\partial^2 \psi_m}{\partial \zeta^2} + a_{55}\omega_{*0m}^2 \psi_m = 0 \tag{3.9}$$

Substituting expansions (3.8) into Eq. (3.2), multiplying both sides of the equality obtained by $\{\psi_k(\zeta)\}$, integrating in the limits $-1 \le \zeta \le 1$ and taking Eq. (3.9) into account we obtain

$$\sum b_{1nm} a_{55}((\omega_{*0n}^u)^2 - (\omega_{*0m}^u)^2)\delta_{mk} + 2a_{55}\omega_{*0n}^u \omega_{*1n}^u C_{2n}^{(0,u)}\delta_{nk} = \sum f_{1m}^u \delta_{mk} \tag{3.10}$$

where δ_{mk} is the Kronecker delta.

If $k \neq n$, we can uniquely determine b_{1nk} from relation (3.10)

$$b_{1nk} = \frac{f_{1k}^u}{a_{55}((\omega_{*0n}^u)^2 - (\omega_{*0k}^u)^2)} = \frac{(r_1 + r_2)C_{2k}^{(0,u)}}{\pi^2(k-n)(n+k+1)} \tag{3.11}$$

When $k = n$ we have

$$\omega_{*1n}^u = \frac{f_{1n}^u}{2a_{55}\omega_{*0n}^u C_{2n}^{(0,u)}} = \frac{2f_{1n}^u}{\pi(2n+1)\sqrt{a_{55}}C_{2n}^{(0,u)}} = -\frac{r_1 + r_2}{2\pi(2n+1)\sqrt{a_{55}}} \tag{3.12}$$

Hence, confining ourselves to the first two approximations, we obtain

$$\omega_{*n}^u = \omega_{*0n}^u + \varepsilon\omega_{*1n}^u = \frac{\pi^2(2n+1)^2 - 2\varepsilon(r_1 + r_2)}{4\pi(2n+1)\sqrt{a_{55}}} \tag{3.13}$$

In the first expansion of (3.8) the value of the coefficient b_{1nm} remains undetermined. To find it we will use the normalization condition[22,23]

$$\int_{-1}^{1}(u_n^{(0,u)} + \varepsilon u_n^{(1,u)})^2 d\zeta \left[\int_{-1}^{1}(u_n^{(0,u)})^2 d\zeta\right]^{-1} = 1$$

whence we have the equality

$$\int_{-1}^{1} u_n^{(0,u)} u_n^{(1,u)} d\zeta = 0$$

and substituting into this the first expansion of (3.8), we obtain

$$b_{1nn}(\xi, \eta) = 0$$

Hence, we have

$$u_n^{(1,u)}(\xi, \eta, \zeta) = \sum \frac{C_{2m}^{(0,u)}(\xi, \eta)(r_1 + r_2)}{\pi^2(m-n)(n+m+1)} \cos\frac{\pi(2m+1)}{4}(1-\zeta) \tag{3.14}$$

The cases $\omega_{*n} = \omega_{*n}^v$ and $\omega_{*n} = \omega_{*n}^w$ can be considered in the same way as the previous one.

If we confine ourselves to the approximations $s = 0$, 1, we will have for the frequencies ω_{*n}^v and ω_{*n}^w expressions similar to formula (3.13) with $\sqrt{a_{55}}$ replaced by $\sqrt{a_{44}}$ and $\sqrt{\Delta/\Delta_{12}}$ in it respectively.

Shear natural vibrations of the shell correspond to the frequencies ω_{*n}^v and ω_{*n}^v while longitudinal vibrations correspond to the frequencies ω_{*n}^w. Since, by formula (2.12), the values of the frequencies for $s = 0$ are identical with the corresponding frequencies for orthotropic plates, the effect of the shell, i.e. the effect of the curvature of the middle surface, manifests itself beginning with the approximation $s = 1$ and is due to terms proportional to $(r_1 + r_2)$. Note that, for orthotropic plates, the effect of the last approximations on the value of the frequencies will be of the order of ε^2.

In a similar way we can consider the approximations $s \geq 2$. However, they are hardly of any interest for applications.

4. The natural vibrations of an orthotropic cylindrical shell

We will consider the special case when the orthotropic shell is cylindrical ($r_1 = 0$, $r_2 = 1$, $A = B = 1$, $k_\alpha = k_\beta = 0$). For the zeroth approximation, relations (2.12) and solution (2.13) remain unchanged. Taking the first approximation for the frequencies of the natural vibrations into account, we have

$$\omega^{u}_{*n} = \omega^{u}_{*0n} + \varepsilon\omega^{u}_{*1n} = \frac{\alpha_n}{\sqrt{a_{55}}}\left(u, \upsilon, w; \sqrt{a_{55}}, \sqrt{a_{44}}, \sqrt{\frac{\Delta}{\Delta_{12}}}\right)$$
$$\alpha_n = \frac{\pi^2(2n+1)^2 - 2\varepsilon}{4\pi(2n+1)} \tag{4.1}$$

Formulae (2.13) and the formulae

$$\upsilon_n^{(0,u)} = w_n^{(0,u)} = 0, \quad u_n^{(0,\upsilon)} = w_n^{(0,\upsilon)} = 0, \quad u_n^{(0,w)} = \upsilon_n^{(0,w)} = 0, \quad \upsilon_n^{(1,u)} = 0$$
$$w_n^{(1,u)} = C_{5n}^{(1,u)}\sin\sqrt{\frac{\Delta}{\Delta_{12}}}\omega^{u}_{*0n}\zeta + C_{6n}^{(1,u)}\cos\sqrt{\frac{\Delta}{\Delta_{12}}}\omega^{u}_{*0n}\zeta + w_0^{(1,u)} \ (u, \upsilon) \tag{4.2}$$
$$u_n^{(1,u)}(\xi,\eta,\zeta) = \sum\frac{C_{2m}^{(0,u)}(\xi,\eta)}{\pi^2(m-n)(n+m+1)}\psi_m(\zeta) \ (u, \upsilon, w; 2m, 4m, 6m)$$

for the components of the displacement vector remain true, where $w_0^{(1,u)}$ is the particular solution of the corresponding equation when $s=1$.

5. Natural vibrations for other boundary conditions

Using solutions (2.7) and (2.11), by satisfying the boundary conditions (1.4) and (1.5) when $s=0$, we obtain two possible versions of the values of the frequencies of the natural vibrations

$$\omega^{uI}_{*0n} = \frac{\pi n}{\sqrt{a_{55}}}, \quad \omega^{uII}_{*0n} = \frac{\pi(n+1/2)}{\sqrt{a_{55}}}, \quad n \in N\left(u, \upsilon, w; \sqrt{a_{55}}, \sqrt{a_{44}}, \sqrt{\frac{\Delta}{\Delta_{12}}}\right) \tag{5.1}$$

with eigenfunctions $\varphi_{1n} = \sin\pi n\zeta$ and $\varphi_{2n} = \cos\pi(n+1/2)\zeta$ respectively.

The frequencies (2.12), corresponding to $\sqrt{a_{55}}$, $\sqrt{a_{44}}$, and also (5.1) corresponding to $\sqrt{\Delta/\Delta_{12}}$ satisfy conditions (1.4) and (1.6), while the eigenfunctions will be φ_{1n}, φ_{2n} and $\varphi_{3n} = \sin\pi(n/2+1/4)(1-\zeta)$.

The approximations $s \geq 1$ are constructed in the same way. Confining ourselves to the approximations $s=0, 1$, we conclude that the following frequencies will correspond to the conditions (1.4) and (1.5)

$$\omega^{uI}_{*n} = \omega^{uI}_{*0n} + \varepsilon\omega^{uI}_{*1n} = \frac{\pi n(4+\varepsilon(r_1+r_2))}{4\sqrt{a_{55}}}\left(u, \upsilon, w; \sqrt{a_{55}}, \sqrt{a_{44}}, \sqrt{\frac{\Delta}{\Delta_{12}}}\right) \tag{5.2}$$

$$\omega^{uII}_{*n} = \omega^{uII}_{*0n} + \varepsilon\omega^{uII}_{*1n} = \frac{4\pi^2(2n+1)^2 + (4+\pi^2(2n+1)^2)\varepsilon(r_1+r_2)}{8\pi(2n+1)\sqrt{a_{55}}}$$
$$\left(u, \upsilon, w; \sqrt{a_{55}}, \sqrt{a_{44}}, \sqrt{\frac{\Delta}{\Delta_{12}}}\right) \tag{5.3}$$

while the frequencies

$$\omega^{u}_{*n} = \omega^{u}_{*0n} + \varepsilon\omega^{u}_{*1n} = \frac{\pi^2(2n+1)^2 + 2\varepsilon(r_1+r_2)}{4\pi(2n+1)\sqrt{a_{55}}}\ (u, \upsilon; \sqrt{a_{55}}, \sqrt{a_{44}}) \tag{5.4}$$

and the frequencies ω^{w}_{*n}, which are identical with (5.2) or (5.3), will correspond to the boundary conditions (1.4) and (1.6).

Other combinations of the conditions (1.3)–(1.6) can be considered in the same way.

6. Natural vibrations in the boundary layer

In order to investigate the natural vibrations in the boundary layer in the region of the side surface $\alpha = \alpha_0$, we will consider the dimensionless components of the displacement vector

$$U = Ru, \quad V = R\upsilon, \quad W = Rw$$

and we will introduce new independent variables by the formulae

$$\alpha - \alpha_0 = h\xi, \quad \beta = R\eta, \quad \gamma = h\zeta$$

Carrying out the procedure for constructing the boundary layer described previously in Ref. 9, the solution of the converted system of Eqs. (1.1), (1.2) will be sought in the form (2.1), (2.3), giving all the required quantities the subscript b. As a result, we obtain the following system for $Q_b^{(s)}$:

$$A_0\frac{\partial\tau_{11b}^{(s)}}{\partial\xi} + \frac{\partial\tau_{13b}^{(s)}}{\partial\zeta} + c^{(j)}u_b^{(s-j)} = R_{1\tau}^{(s-1)} \; (11b, 12b, 13b;\; 13b, 23b, 33b;\; u, \upsilon, w;\; 1\tau, 2\tau, 3\tau)$$

$$A_0\frac{\partial u_b^{(s)}}{\partial\xi} - \sum_{1b}^{(s)} = R_u^{(s-1)}, \quad \sum_{2b}^{(s)} = R_\upsilon^{(s-1)}, \quad \frac{\partial w_b^{(s)}}{\partial\zeta} - \sum_{3b}^{(s)} = R_w^{(s-1)}$$

$$\frac{\partial\upsilon_b^{(s)}}{\partial\zeta} - a_{44}\tau_{23b}^{(s)} = R_{4\tau}^{(s-1)}, \quad A_0\frac{\partial w_b^{(s)}}{\partial\xi} + \frac{\partial u_b^{(s)}}{\partial\zeta} - a_{55}\tau_{13b}^{(s)} = R_{5\tau}^{(s-1)} \qquad (6.1)$$

$$A_0\frac{\partial\upsilon_b^{(s)}}{\partial\xi} - a_{66}\tau_{12b}^{(s)} = R_{6\tau}^{(s-1)}, \quad \tau_{12b}^{(s)} - \tau_{21b}^{(s)} = R_{7\tau}^{(s-1)}$$

$$A_0 = A(\alpha_0), \quad c^{(j)} = \sum_{n=0}^{j}\omega_{*(j-n)}\omega_{*(n)}, \quad j = \overline{0, s}, \quad \sum_{ib}^{(s)} = a_{i1}\tau_{11b}^{(s)} + a_{i2}\tau_{22b}^{(s)} + a_{i3}\tau_{33b}^{(s)}$$

where $R_{i\tau}^{(s-1)}$ are functions that are known for each approximation if we know the values of the preceding approximations, in particular $R_{i\tau}^{(k)} \equiv 0$ when $k < 0$.

From system (6.1) the components of the stress tensor can be expressed in terms of $u_b^{(s)}$, $\upsilon_b^{(s)}$, $w_b^{(s)}$:

$$\tau_{23b}^{(s)} = \frac{1}{a_{44}}\left[\frac{\partial\upsilon_b^{(s)}}{\partial\zeta} - R_{4\tau}^{(s-1)}\right], \quad \tau_{12b}^{(s)} = \frac{1}{a_{66}}\left[A_0\frac{\partial\upsilon_b^{(s)}}{\partial\xi} - R_{6\tau}^{(s-1)}\right], \quad \tau_{12b}^{(s)} - \tau_{21b}^{(s)} = R_{7\tau}^{(s-1)}$$
$$\tau_{13b}^{(s)} = \frac{1}{a_{55}}\left[A_0\frac{\partial w_b^{(s)}}{\partial\xi} + \frac{\partial u_b^{(s)}}{\partial\zeta} - R_{5\tau}^{(s-1)}\right] \qquad (6.2)$$

$$\tau_{11b}^{(s)} = \frac{1}{\Delta}\left[\left(A_0\frac{\partial u_b^{(s)}}{\partial\xi} - R_u^{(s-1)}\right)\Delta_{23} + R_\upsilon^{(s-1)}\Delta_1 + \left(\frac{\partial w_b^{(s)}}{\partial\zeta} - R_w^{(s-1)}\right)\Delta_2\right] \qquad (6.3)$$
$$(11b, 22b, 33b;\; \Delta_{23}, \Delta_1, \Delta_2;\; \Delta_1, \Delta_{13}, \Delta_3;\; \Delta_2, \Delta_3, \Delta_{12})$$

while to determine the components of the displacement vector we obtain the equations

$$\frac{1}{a_{66}}A_0^2\frac{\partial^2\upsilon_b^{(s)}}{\partial\xi^2} + \frac{1}{a_{44}}\frac{\partial^2\upsilon_b^{(s)}}{\partial\zeta^2} + c^{(j)}\upsilon_b^{(s-j)} = T_\upsilon^{(s-1)}, \quad j = \overline{0, s} \qquad (6.4)$$

$$\frac{\Delta_{23}}{\Delta}A_0^2\frac{\partial^2 u_b^{(s)}}{\partial\xi^2}+A_0\left(\frac{\Delta_2}{\Delta}+\delta_1\right)\frac{\partial^2 w_b^{(s)}}{\partial\xi\partial\zeta}+\frac{1}{a_{55}}\frac{\partial^2 u_b^{(s)}}{\partial\zeta^2}+c^{(j)}u_b^{(s-j)} = T_u^{(s-1)}$$
$$\left(u, w; \frac{\Delta}{\Delta_{23}}, a_{55}; a_{55}, \frac{\Delta}{\Delta_{12}}\right) \tag{6.5}$$

where

$$\delta_1 = \frac{1}{a_{55}},\quad T_v^{(s-1)} = R_{2\tau}^{(s-1)}+\frac{1}{a_{66}}A_0\frac{\partial R_{6\tau}^{(s-1)}}{\partial\xi}+\frac{1}{a_{44}}\frac{\partial R_{4\tau}^{(s-1)}}{\partial\zeta}$$
$$T_u^{(s-1)} = R_{1\tau}^{(s-1)}+\frac{\Delta_{23}}{\Delta}A_0\frac{\partial R_u^{(s-1)}}{\partial\xi}+\frac{\Delta_2}{\Delta}A_0\frac{\partial R_w^{(s-1)}}{\partial\xi}+\frac{1}{a_{55}}\frac{\partial R_{5\tau}^{(s-1)}}{\partial\zeta}-\frac{\Delta_1}{\Delta}A_0\frac{\partial R_v^{(s-1)}}{\partial\xi} \tag{6.6}$$
$$\left(u\leftrightarrow w; A_0\frac{\partial}{\partial\xi}\leftrightarrow\frac{\partial}{\partial\zeta}; 1\tau, 3\tau; \Delta_{23}, \Delta_{12}; \Delta_1, \Delta_3\right)$$

Eq. (6.4) and relations (6.2) describe an antiplane boundary layer while (6.3) and (6.5) describe a plane boundary layer.

The initial approximation is of particular interest. When $s=0$ the right-hand sides of Eqs. (6.4) and (6.5) vanish. It is necessary to obtain an attenuating solution of Eq. (6.4) with the boundary conditions

$$\zeta = -1:\ v_b^{(0)} = 0,\quad \zeta = 1:\ \tau_{23b}^{(0)} = 0 \tag{6.7}$$

and systems of Eq. (6.5) with the conditions

$$\zeta = 1:\ \tau_{13b}^{(0)} = \tau_{33b}^{(0)} = 0,\quad \zeta = -1:\ u_b^{(0)} = w_b^{(0)} = 0 \tag{6.8}$$

The solution of Eq. (6.4) with $s=0$ will be sought in the form

$$v_b^{(0)}(\zeta,\eta,\zeta) = \exp(-\lambda_a\xi)C^{(0)}(\eta)v_{1b}^{(0)}(\zeta) \tag{6.9}$$

The subscript a denotes that λ_a belongs to the antiplane boundary layer. Substituting expression (6.9) into Eq. (6.4), we obtain a homogeneous ordinary differential equation, the general solution of which has the form

$$v_{1b}^{(0)}(\zeta) = C_1^{(0)}\sin\alpha_a\zeta+C_2^{(0)}\cos\alpha_a\zeta,\quad \alpha_a = \sqrt{a_{44}\left(\omega_{*0}^2+\frac{A_0^2}{a_{66}}\lambda_a^2\right)}$$

Satisfying boundary conditions (6.7), we obtain

$$\cos 2\sqrt{a_{44}\left(\omega_{*0}^2+\frac{A_0^2}{a_{66}}\lambda_a^2\right)} = 0 \Rightarrow \lambda_{ank} = \pm\sqrt{\frac{a_{66}}{A_0^2}\left(\frac{\pi^2(2n+1)^2}{16a_{44}}-\omega_{*0k}^2\right)} \tag{6.10}$$

By virtue of the property of the boundary layer it is necessary to confine ourselves to the values λ_{ank} with $\mathrm{Re}\lambda_{ank}>0$. The eigenfunctions will be

$$v_{bnk}^{(0)}(\xi,\eta,\zeta) = C^{(0)}(\eta)\exp(-\lambda_{ank}\xi)\cos\frac{\pi(2n+1)(1-\zeta)}{4} \tag{6.11}$$

The solution of system (6.5) with $s=0$ will be sought in the form

$$u_b^{(0)}(\xi,\eta,\zeta) = K_b^{(0)}(\eta)\exp(-\lambda_p\xi+k\zeta)$$
$$w_b^{(0)}(\xi,\eta,\zeta) = LK_b^{(0)}(\eta)\exp(-\lambda_p\xi+k\zeta) \tag{6.12}$$

where L is an as yet undetermined multiplier and k is the root of the characteristic equation

$$B_2k^4 + (\lambda_p^2B_3 + B_5)k^2 + \lambda_p^4B_1 + \lambda_p^2B_4 + \omega_{*0}^4 = 0$$

where

$$B_1 = \frac{\Delta_{23}}{\Delta a_{55}}A_0^4, \quad B_2 = \frac{\Delta_{12}}{\Delta a_{55}}, \quad B_3 = \left(\frac{\Delta_{23}\Delta_{12} - \Delta_2^2}{\Delta^2} - 2\frac{\Delta_2}{\Delta a_{55}}\right)A_0^2$$

$$B_4 = \left(\frac{\Delta_{23}}{\Delta} + \frac{1}{a_{55}}\right)A_0^2\omega_{*0}^2, \quad B_5 = \left(\frac{\Delta_{12}}{\Delta} + \frac{1}{a_{55}}\right)\omega_{*0}^2$$

Hence it follows that

$$k_{1,2}^2 = (-\lambda_p^2B_3 - B_5 \pm \sqrt{D})/(2B_2)$$
$$D = \lambda_p^4(B_3^2 - 4B_1B_2) + 2\lambda_p^2(B_3B_5 - 2B_2B_4) + B_5^2 - 4B_2\omega_{*0}^4 \quad (6.13)$$

Each k_i has its own factor L_i.

Substituting expressions (6.12) into system (6.5), we obtain

$$L_i = \frac{\Delta_{23}a_{55}\lambda_p^2A_0^2 + \Delta k_i^2 + \Delta a_{55}\omega_{*0}^2}{(\Delta + \Delta_2a_{55})\lambda_pk_i} \quad (6.14)$$

As a result, the solution of system (6.5) with $s = 0$ takes the form

$$u_b^{(0)}(\xi, \eta, \zeta) = \sum_{i=1}^{4}K_{ib}^{(0)}(\eta)\exp(-\lambda_p\xi + k_i\zeta)$$
$$w_b^{(0)}(\xi, \eta, \zeta) = \sum_{i=1}^{4}L_iK_{ib}^{(0)}(\eta)\exp(-\lambda_p\xi + k_i\zeta) \quad (6.15)$$

Substituting expressions (6.15) into conditions (6.8) and taking relations (6.3) into account, we obtain a system of homogeneous algebraic equations, for a non-trivial solution of which to exist the following equality must be satisfied

$$\sum_{(1,2,3,4)}(-1)^1S_1[Q_2(L_3 - L_4) + Q_3(L_4 - L_2) + Q_4(L_2 - L_3)] = 0 \quad (6.16)$$

where

$$S_i = (\Delta_{12}k_iL_i - \Delta_2\lambda_pA_0)\exp(2k_i), \quad Q_i = (k_i - \lambda_pA_0L_i)\exp(2k_i), \quad i = 1, 2, 3, 4 \quad (6.17)$$

The summation on the left-hand side of Eq. (6.16) is carried out over a cyclic rearrangement of the indices, taking into account alternation in the signs of the terms.

Eq. (6.16) is the characteristic equation for determining λ_p.

In relations (6.10) and (6.16), ω_{*0} occurs as a parameter, and a denumerable set of λ_a and λ_p will correspond to each of its values, defined by Eq. (2.12). Hence, for each eigenvalue there is a family of boundary functions. Hence, for example, both λ_a^u and λ_p^u will correspond to the frequency ω_{*0n}^u, i.e. natural vibrations of one type generate vibrations of the other type in the boundary layer.

In the upper part of Table 1 we show the first of several values of λ_a for a shell of plexiglass 2:1 with the following characteristics

$$\nu_{12} = 0.105, \quad \nu_{23} = 0.431, \quad \nu_{31} = 0.405$$

$$E_1 = 36 \cdot 10^3 \text{ MPa}, \quad E_2 = 26.3 \cdot 10^3 \text{ MPa}, \quad E_3 = 10.8 \cdot 10^3 \text{ MPa}$$

$$G_{12} = 4.9 \cdot 10^3 \text{ MPa}, \quad G_{23} = 4 \cdot 10^3 \text{ MPa}, \quad G_{13} = 4.4 \cdot 10^3 \text{ MPa}$$

Table 1

k	$\omega_{*0k}=\omega^{u}_{*0k}$	$\omega_{*0k}=\omega^{v}_{*0k}$	$\omega_{*0k}=\omega^{w}_{*0k}$
Antiplane boundary layer			
0	1.995, 3.469, 4.911, 6.343, 7.770, 9.195	2.007, 3.476, 4.916, 7.347, 7.773, 9.198	0.688, 2.921, 4.540, 6.060, 7.541, 9.002
1	2.757, 4.437, 5.984, 7.480, 8.951, 10.407	2.838, 4.488, 6.02, 7.510, 8.976, 10.429	2.063, 4.939, 6.969, 8.762, 10.440, 12.052
2	3.290, 5.190, 6.862, 8.441, 9.973, 11.475	3.476, 5.310, 6.953 8.515, 10.035, 11.530	3.439, 6.637, 8.961, 10.981, 12.842, 14.603
3	3.694, 5.813, 7.613, 9.282, 10.880, 12.436	4.014, 6.021, 7.773, 9.414, 10.993, 12.534	4.814, 8.215, 10762, 12.969, 14.987, 16.883
4	4.008, 6.343, 8.272, 10.033, 11.701, 13,312	4.488, 6.657, 8.515, 10.234, 11.874, 13.464	6.1893, 9.732, 12.455, 14.819, 16.973, 18.989
5	4.252, 6.803, 8.860, 10.713, 12.452, 14.119	4.916, 7.237, 9.198, 10.993, 12.694, 14.334	7.565, 11.212, 14.079, 16.576, 18.850, 20.9
Plane boundary layer			
0	$0.264+0.243i$, $0.853+0.602i$, 1.966, 2.454, $2.953+0.4796i$, 4.194	$0.252+0.231i$, $0.868+0.596i$, 1.963, 2.454, $2.959+0.481i$, 4.191	$0.715+0.657i$, $2.091+0.589i$, 2.436, 2.850, $4.295+0.436i$, 4.635
1	$0.793+0.728i$, $1.947+0.688i$, 2.428, 2.888, $4.210+0.503i$, 4.635	0.074, $0.756+0.694i$, $2.016+0.644i$, 2.432, 2.873, $4.251+0.473i$	1.689, 2.110, $2.146+1.971i$, $3.984+0.758i$, 5.038, 5.559
2	0.409, $1.321+1.213i$, 2.331, 2.852, $3.427+0.677i$, 4.659	$0.661+0.563i$, $1.260+1.157i$ 2.347, 2.872, $3.546+0.689i$, 4.648	2.278, $3.397+0.325i$, 4.626, 5.434, $6.007+0.581i$, 7.798
3	0.683, $1.850+1.699i$, 2.249, 2.407, 3.080, $4.551+0.693i$	0.861, 1.026 $1.764+1.6195i$, 2.276, 2.5273, 3.093	0.636, 2.460, $4.145+0.614i$, $5.738+0.656i$, $7.459+0.410i$, 7.927
4	$2.253+0.096i$, $3.619+0.680i$, 5.032, 5.509, $6.057+0.676i$, 7.330	1.0097, 2.114, 2.633, $3.7798+0.723i$, 5.027, 5.537	$2.776+0.543i$ $4.950+0.359i$, $6.485+0.288i$, $7.922+0.576i$ $9.514+0.678i$, 10.453

while in the lower part we show the first few values of λ_p for a plane boundary layer in the case of a cylindrical orthotropic shell of the same material; the values $\omega^{u}_{*0k}(u, v, w)$ are given by formula (2.12). It can be seen that the real parts of the exponents increase in absolute value fairly rapidly, and in applied calculations one can confine oneself to the first few roots of Eqs. (6.10) and (6.16).

Note that when $s=0$ the values of λ_a for an antiplane boundary layer in the case of a cylindrical orthotropic shell are identical with the values of λ_a for an orthotropic plate of the same material.[18]

It follows from the data in the table that for natural vibrations the attenuation in an antiplane boundary layer is more rapid than in a plane boundary layer.

The approximations $s\geq 1$ for a boundary layer can be constructed in a similar way, but it is not of much interest from the applications point of view.

References

1. Gol'denveizer AL. *The Theory of Thin Elastic Shells*. Moscow: Nauka; 1976.
2. Gol'denveizer AL. Algorithms for the asymptotic construction of a two-dimensional theory of thin shells and the St Venant principle. *Prikl Mat Mekh* 1995;**59**(6):1019–32.
3. Gol'denveizer AL. Internal and boundary calculations of thin elastic solids. *Prikl Mat Mekh* 1995;**59**(6):1019–32.
4. Gol'denveizer AL. Approximate methods of analysing thin elastic shells and plates. *Izv Ross Akad Nauk MTT* 1997;**3**:134–49.

5. Gol'denveizer AL, Lidskii VB, Tovstik PYe. *Free Vibrations of Thin Elastic Shells*. Moscow: Nauka; 1979.
6. Kaplunov YuD. Integration of the dynamic boundary-layer equations. *Izv Akad Nauk SSSR MTT* 1990;**1**:148–60.
7. Kaplunov YuD. Vibrations of shells of revolution with a high-frequency excitation boundary. *Izv Akad Nauk SSSR MTT* 1991;**6**:151–9.
8. Kaplunov YuD, Kirillova IV, Kossovich LYu. Asymptotic integration of the dynamic equations of the theory of elasticity for the case of thin shells. *Prikl Mat Mekh* 1993;**57**(1):83–91.
9. Agalovyan LA. *Asymptotic Theory of Anisotropic Plates and Shells*. Moscow: Nauka; 1997.
10. Agalovyan LA. The structure of the solution of a class of plane problems of the theory of elasticity. In *Mechanics Issue* 2. Izd. Yerevan. Univ., Yerevan, 1982, 7–12.
11. Agalovyan LA. Determination of the stress-strain state of a two-layer strip and the correctness of Winkler's hypothesis. In Proceedings of the 13th All-Union Conference on the Theory of Plates and Shells. Pt.1 Tallin, 1983, 13–18.
12. Agalovyan LA, Gevorkyan RS. Non-classical boundary-value problems of plates with general anisotropy. In Proceedings of the 4th Symposium on the Mechanics of Structures of Composite Materials. Nauka, Novosibirsk, 1984, 105–10.
13. Agalovyan LA, Gevorkyan RS. The asymptotic solution of mixed three-dimensional problems for double-layer anisotropic plates. *Prikl Mat Mekh* 1986;**50**(2):271–8.
14. Agalovyan LA, Gevorkyan RS. The asymptotic solution of non-classical boundary-value problems of double-layer anisotropic thermoelastic shells. *Izv Akad Nauk ArmSSR Mekhanika* 1989;**42**(3):28–36.
15. Agalovyan LA, Sarkisyan LS. The natural vibrations of a double-layer orthotropic strip. In Proceedings of the 18th International Conference on the Theory of Shells and Plates. Vol. 1, Saratov, 1997, 30–8.
16. Agalovyan ML. An eigenvalue problem that occurs in seismology. *Dokl Nats Akad Nauk Armenii* 1996;**96**(24):23–8.
17. Agalovyan LA. The asymptotic method of solving dynamic mixed problems of anisotropic strips and plates. *Izv Vuzov Severo-Kavkaz Region Yestestv Nauki* 2000;**3**:8–11.
18. Agalovyan LA, Gulgazaryan LG. The frequencies of natural vibrations and the boundary layer for an orthotropic plate in the mixed boundary-value problem. *Izv Nats Akad Nauk Resp Armenii Mekhanika* 2001;**54**(2):32–41.
19. Agalovyan LA. A class of problems of the forced vibrations of anisotropic plates. In *Problems of the Mechanics of Thin Deformable Bodies*. Izd Nats Akad Nauk Resp Armenii Yerevan, 2002, 9–19.
20. Agalovyan LA. The asymptotic form of the solutions of classical and non-classical boundary-value problems of the statics and dynamics of thin bodies. *Prikl Mekhanika* 2002;**38**(7):3–24.
21. Green AE. On the linear theory of thin elastic shells. *Proc Roy Soc London Ser A* 1962;**266**(1325):143–60.
22. Nayfeh A. *Perturbation Methods*. New York: Wiley; 1973.
23. Lomov SA. *Introduction to the General Theory of Singular Perturbations*. Moscow: Nauka; 1981.

Translated by R.C.G.